Spontaneous chiral symmetry breaking in QCD

Spezialisierungsmodul

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1 Introduction

The theory, which describes the strong interaction between hadrons and their constituents is called Quantumchromodynamics, or QCD for short. On a quantum field theory level, its main components are N_f massive fermion fields, called quarks, and eight $SU(3)$ gauge fields, called gluons. When taking a look at the spectrum of light hadrons and comparing it to the masses of the bare quarks, there are a few things that seem odd at a first glance. First of all, it seems odd, that the masses of the light hadrons are so much higher than the bare masses of their quark constituents. In fact, the masses of the proton and neutron are about 100 times larger than the combined masses of their quark content [1]. Knowing this, it should come as a surprise, that the light pseudoscalar mesons, in particular the pions, kaons and the η, are considerably lighter compared to the light baryons.
These phenomena are consequences of the symmetries of the underlying theory. We want to focus on a particular effect called spontaneous symmetry breaking. It is the process, which is responsible for the light masses of the pseudoscalar mesons. It also makes the quarks obtain an effective mass due to self interaction, which is much higher than their bare mass.

2 Symmetries

To understand the concept of spontaneous symmetry breaking, it is useful to first of all think about what a symmetry actually is. For our purposes we will use the following definition:

A symmetry is a transformation, which when applied to the fundamental degrees of freedom of a theory leaves the action unchanged.

In field theories, such as QCD, the fundamental degrees of freedom mentioned above are the fields themselves. In the case of QCD they are the quark and gluon fields. The transformations can be split into two different kind of transformations, discrete and continuous ones. Discrete transformations are classified by the fact, that they form a group, which has a countable amount of elements. These include, but are not limited to, spatial inversion, permutation and charge conjugation. Continuous transformations also form a group, but in contrast to discrete transformations the group has an uncountable amount of elements, which can be continuously transformed into each other by varying a set of parameters. Such transformations include Poincaré transformations, $U(N)$ transformations and $SO(N)$ transformations. Next we shall see how symmetries actually affect the physics of a theory.

2.1 Consequences of symmetries

On top of oftentimes simplifying calculations, symmetries also manifest themselves in the actual physics of a theory. The most notable consequence of a continuous symmetry is Noether's theorem. It states that for every continuous symmetry of a field theory there is a conserved current $\partial_\mu j^\mu = 0$. An important example for this is that space-time symmetries, i.e. invariance under spatial rotations and space-time translations, imply the conservation of energy, momentum and angular momentum. Other symmetries also have conservation laws attached to them, some of which we will take a closer look at later.

2.2 Symmetry breaking

Since symmetries play an important role in our understanding of filed theories, it is also interesting to look at symmetries, that are not exact, but broken in some way.

Explicit symmetry breaking

The most obvious way to break a symmetry is by adding a term to the action, which is not invariant under the symmetry transformation. One such case is a mass term breaking the axial $U(1)_A$ and $SU(N)_A$ symmetries of a theory with N flavours of fermions in it. This is also the case in QCD, as we will see later. Symmetries do not have to be broken explicitly though. They can be broken in different, more subtle ways.

Anomalous symmetry breaking

One such way is via anomalous symmetry breaking. It only occurs in quantum field theories, when quantum fluctuations break an otherwise fine symmetry of the classical action. This manifests itself through the fact, that such symmetries are broken by the introduction of an ultraviolet regulator and that the symmetry is not restored when the regulator is removed after renormalization. This results in a non-zero divergence of the current j^μ, which does not vanish when the regulator is removed, so the current is not conserved. A famous example for this is the $U(1)_A$ symmetry of a gauge theory with fermions [2]. As mentioned above, a mass term breaks this symmetry explicitly, but even if the fermions are massless, the symmetry is broken anomalously.

Spontaneous symmetry breaking

Another way to break a symmetry is spontaneous symmetry breaking. We call a symmetry spontaneously broken, if it is a symmetry of the action, but not of the ground state of the theory. Note, that this is not the same as anomalous symmetry breaking. While anomalies occur due to quantum fluctuations and the need of a regulator in quantum field theories, spontaneous symmetry breaking can also appear outside of quantum field theories. The prime example for this is a ferromagnet under the critical temperature T_c. At these temperatures the ground state of the ferromagnet has a non zero magnetization along some axis. Therefore, the $SO(3)$ symmetry of the action is broken down to an $SO(2)$ symmetry, representing rotations along its magnetization axis.

Spontaneous symmetry breaking is particularly interesting, since it has direct consequences on physical observables. Most notably, every spontaneously broken continuous symmetry leads to one massless boson appearing in the spectrum of the theory. This is called Goldstone's theorem, the massless bosons are called Goldstone bosons.

While this theorem can be proven in general [3], it is helpful to find a way to intuitively think about this result. If we assume a theory with a spontaneously broken continuous symmetry, this theory will have a continuum of degenerate ground states, which can be transformed into each other via that symmetry transformation. Thus, the ground states can be labelled by a set of parameters corresponding to the parameters of the transformation. We will write them as θ. Given a ground state $|\theta\rangle$, a transformation into a different ground state $|\theta'\rangle$ requires no additional energy. Therefore, it can be transmitted by excitations with arbitrary small energies. This implies, that there is no mass gap in the spectrum, so there have to be massless particles, which correspond to these excitations.

This already hints at the low masses of the pseudoscalar mesons in the QCD spectrum. Later we will see, that they are in fact Goldstone bosons of a spontaneously broken symmetry. But in order to understand, which symmetry this is and why their masses are not exactly zero in reality, we have to take a closer look at the symmetries of QCD.

2.3 Symmetries of QCD

We now want to investigate the symmetries of QCD. Therefore, we take a look at the QCD action, which is the simplest local $SU(3)$ gauge theory with fermions we can write down. In 4-dimensional euclidean space-time it is given by [2]

$$S_{\text{QCD}} = \int \mathrm{d}^4x \left[\sum_{i=1}^{N_f} \overline{\psi}_i \left(\not{D} + m_i \right) \psi_i + \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \right]. \tag{1}$$

Here the ψ_i and $\overline{\psi}_i$ are the quark fields and their conjugate fields, $\not{D} = \not{\partial} + ig\not{A}$ is the covariant derivative and $F^a_{\mu\nu}$ is the gluon field strength tensor. By construction, this action is invariant under a local $SU(3)$ gauge transformation. This action also has a set of global symmetries. We want to focus on the symmetries of the quark fields.
First of all, the action is also invariant under a global $U(1)_V$ transformation of the quark fields

$$\psi_i(x) \to e^{i\theta} \psi_i(x), \tag{2}$$

where θ is a real parameter. The conservation law connected with this symmetry is the conservation of baryon number, which is a fine symmetry even in the full quantum field theory. If the quark masses m_i are zero, the action also has an axial $U(1)_A$ symmetry, which differs from the $U(1)_V$ symmetry by insertion of a γ^5 matrix in the argument of the exponential

$$\psi_i(x) \to e^{i\gamma^5\theta} \psi_i(x). \tag{3}$$

This symmetry is explicitly broken by the quark masses, since the mass term $m_i\overline{\psi}_i\psi_i$ is not invariant under this transformation but picks up a factor of $\exp\left(2i\gamma^5\theta\right)$. Even in the chiral limit when the quark masses are set to zero, the symmetry is broken anomalously as mentioned in the previous section.
The QCD action has additional symmetries under the condition, that there are $N \leq N_f$ flavours of quarks, which have the same mass m_c. In that case, the action is also invariant under an $SU(N)$ transformation which mixes these flavours

$$\psi_i(x) \to \left(U_l\right)_{ij} \psi_j(x), \tag{4}$$

with $l \in \{V, A\}$ labeling the symmetry as vector or axial symmetry. $U_V = \exp\left(i\theta^a\tau^a\right)$ is an $SU(N)_V$ matrix, the τ^a are the $N(N-1)/2$ generators of the $SU(N)$ group and the θ^a are the corresponding parameters. If additionally $m_c = 0$, there is also the axial $SU(N)_A$ symmetry, which again corresponds to an insertion of γ^5 in the argument of the exponential, $U_A = \exp\left(i\gamma^5\theta^a\tau^a\right)$.
In total these global symmetries can be summarized as

$$SU(N)_V \otimes SU(N)_A \otimes U(1)_V \otimes U(1)_A, \tag{5}$$

In the real world, the quark masses are all different and nonzero, therefore all of the symmetries except for the $U(1)_V$ are broken explicitly by mass terms. If the quark masses and their differences are small compared to Λ_{QCD}, which is the case for up-, down- and, arguably, strange quarks these symmetries are good approximate symmetries though.

2.4 The pion as Goldstone boson

In the chiral limit the $SU(N)_A$ symmetry is actually broken spontaneously. By Goldstone's theorem we therefore expect massless Goldstone bosons to appear. In fact, the spontaneous symmetry breaking results in the pion mass being zero for $N = 2$ flavours of massless quarks and the kaon and η having zero mass additionally to the pion for $N = 3$. This can be seen in numerical calculations [4], but it can also be shown analytically. Since the real world does have quark masses, the $SU(N)_A$ symmetry is broken explicitly, therefore the pion mass is not zero. The relation between the pion mass and the quark mass m_c is called the Gell-Mann-Oakes-Renner relation [5]

$$f_\pi^2 m_\pi^2 = -2m_c \left\langle \overline{\psi}\psi \right\rangle /N, \tag{6}$$

with f_π being the decay constant of the pion. $\left\langle \overline{\psi}\psi \right\rangle$ is called the chiral quark condensate. A non-zero value of $\left\langle \overline{\psi}\psi \right\rangle$ indicates, that the chiral $SU(N)_A$ symmetry is spontaneously broken. From this relation we can directly see, that in the chiral limit either the mass or the decay constant of the pion has to be zero. One can further show that if chiral symmetry is spontaneously broken the decay constant has to be non zero [5]. Therefore, the mass of the pion is indeed zero in the chiral limit, making it the Goldstone boson we expect.
Another consequence of the relation 6 is, that the mass of the pion grows proportional to the square root of the quark mass m_c. This explains, why in the real world the pion has a non-zero, but light mass.

References

[1] M. Tanabashi et al. Review of Particle Physics. *Phys. Rev.*, D98(3):030001, 2018.

[2] Mark Srednicki. *Quantum field theory*. Cambridge University Press, 2007.

[3] Jeffrey Goldstone. Field theories with «superconductor» solutions. *Il Nuovo Cimento (1955-1965)*, 19(1):154–164, 1961.

[4] Christian S Fischer. Infrared properties of qcd from dyson–schwinger equations. *Journal of Physics G: Nuclear and Particle Physics*, 32(8):R253, 2006.

[5] Gernot Eichmann, Helios Sanchis-Alepuz, Richard Williams, Reinhard Alkofer, and Christian S Fischer. Baryons as relativistic three-quark bound states. *Progress in Particle and Nuclear Physics*, 91:1–100, 2016.